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Generalized Gaussian beams

E G Abramochkin and V G Volostnikov

Coherent Optics Laboratory, Samara branch of P N Lebedev Physical Institute, Novo Sadovaya 221, Samara 443011, Russia

E-mail: coherent@fian.smr.ru (V G Volostnikov)

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Abstract
A unity of Hermite–Gaussian (HG) and Laguerre–Gaussian (LG) beam families is proposed by introducing an additional parameter. Continuous changing of the introduced parameter allows one to transform HG beams into LG beams in a continuous way, keeping some important properties of both families, for example, structural stability under propagation. The generalized beams (called Hermite–Laguerre–Gaussian beams) are investigated by theoretical and experimental means.

Keywords: Hermite–Gaussian beams, Laguerre–Gaussian beams, mode converter, phase singularities

It is known [1] that Hermite–Gaussian (HG) beams

$$\mathscr{H}_{n,m}(x,y) = \exp(-x^2-y^2)H_n(\sqrt{2}x)H_m(\sqrt{2}y) \qquad (n,m=0,1,\ldots) \tag{1}$$

and Laguerre–Gaussian (LG) beams

$$\mathscr{L}_{n,\pm m}(x,y) = \exp(-x^2-y^2)(x\pm \mathrm{i}y)^m L_n^m(2x^2+2y^2) \qquad (n,m=0,1,\ldots) \tag{2}$$

generate structurally stable solutions of the parabolic equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+2\mathrm{i}k\frac{\partial F}{\partial l}=0. \tag{3}$$

Here k is the wavenumber and l is the propagation variable.

These beam families play an important part in the theory of resonators and optical waveguides. It is also known that every family is a basis for the space $L_2(\mathbb{R}^2)$. Therefore, an arbitrary square integrable two-dimensional function may be presented as a series of HG or LG functions. In particular, any HG function is a linear combination of LG functions and vice versa. In a sense, both families are equivalent. On the other hand, the family symmetries are rather different: namely, HG beams have a rectangular symmetry, while LG beams have a rotational symmetry, and the choice of a convenient family depends on the concrete physical problem.

There are other relations between these two beam families. In [2] some astigmatic transformation of HG beams into LG beams was found and investigated by theoretical and experimental means. The astigmatic transformation may be used to obtain more a general family of coherent light fields, named generalized Gaussian beams, or Hermite–Laguerre–Gaussian beams. This family contains HG and LG beam families as particular representatives.

We start with two results of an astigmatic transformation of HG beams. The first result is a simple one and describes an invariance property of HG beams under the astigmatic influence function $a(\xi^2-\eta^2)$:

$$\begin{aligned}\iint_{\mathbb{R}^2} &\exp\left(-\mathrm{i}(x\xi+y\eta)+\frac{\mathrm{i}a(\xi^2-\eta^2)}{\rho^2}\right) \\ &\times \mathscr{H}_{n,m}\left(\frac{\xi}{\rho},\frac{\eta}{\rho}\right)\mathrm{d}\xi\,\mathrm{d}\eta = \frac{\pi\rho^2(-\mathrm{i})^{n+m}}{\sqrt{1+a^2}} \\ &\times \exp\left(-\frac{\mathrm{i}a\rho^2(x^2-y^2)}{4(1+a^2)}+\mathrm{i}(n-m)\arctan a\right) \\ &\times \mathscr{H}_{n,m}\left(\frac{\rho x}{2\sqrt{1+a^2}},\frac{\rho y}{2\sqrt{1+a^2}}\right).\end{aligned} \tag{4}$$

Another result, which was found in [2], is an astigmatic transformation of HG beams into LG beams with the help of the astigmatic influence function $2\xi\eta$:

$$\begin{aligned}\iint_{\mathbb{R}^2} &\exp\left(-\mathrm{i}(x\xi+y\eta)+\frac{2\mathrm{i}\xi\eta}{\rho^2}\right)\mathscr{H}_{n,m}\left(\frac{\xi}{\rho},\frac{\eta}{\rho}\right)\mathrm{d}\xi\,\mathrm{d}\eta \\ =\ &\frac{\pi\rho^2(-1)^{n+m}}{\sqrt{2}}\exp\left(-\frac{\mathrm{i}\rho^2xy}{4}\right) \\ &\times\begin{cases}(2\mathrm{i})^n m!\,\mathscr{L}_{m,n-m}\left(\dfrac{\rho x}{2\sqrt{2}},\dfrac{\rho y}{2\sqrt{2}}\right) & (n\geqslant m),\\[2ex] (2\mathrm{i})^m n!\,\mathscr{L}_{n,m-n}\left(\dfrac{\rho y}{2\sqrt{2}},\dfrac{\rho x}{2\sqrt{2}}\right) & (n\leqslant m).\end{cases}\end{aligned} \tag{5}$$

There are various ways to realize the transformation (5) by means of cylindrical and spherical optical elements. For example, if we multiply both sides of the equality (5) by $\exp(\mathrm{i}\rho^2 xy/4)$ then we obtain the case of an optical scheme with residual astigmatism compensation. Some experimental set-ups (mode converters) for transforming HG beams into LG beams are presented in [2–5]. The main feature of all converters based on equality (5) is that the angle between the symmetry axis of the HG beams and the elements of the cylindrical lenses is $\alpha = \pi/4$.

For an arbitrary angle α the astigmatic influence function is as follows:

$$\psi(\xi, \eta, \alpha) = (\xi^2 - \eta^2)\cos 2\alpha + 2\xi\eta \sin 2\alpha. \tag{6}$$

Applying this to an HG beam, the result of the corresponding astigmatic transformation is a specific linear combination of HG beams (see the appendix in [2] for the proof):

$$\begin{aligned}
&\int\!\!\int_{\mathbb{R}^2} \exp\left(-\mathrm{i}(x\xi + y\eta) + \frac{\mathrm{i}\psi(\xi,\eta,\alpha)}{\rho^2}\right) \\
&\quad \times \mathscr{H}_{n,m}\left(\frac{\xi}{\rho}, \frac{\eta}{\rho}\right) \mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad = \frac{\pi\rho^2}{\sqrt{2}}\left(\frac{1-\mathrm{i}}{\sqrt{2}}\right)^{n+m} \\
&\quad \times \exp\left(-\frac{\mathrm{i}\rho^2\psi(x,y,\alpha)}{8}\right)\sum_{k=0}^{n+m} \mathrm{i}^k \cos^{n-k}\alpha \sin^{m-k}\alpha \\
&\quad \times P_k^{(n-k,m-k)}(-\cos 2\alpha) \\
&\quad \times \mathscr{H}_{n+m-k,k}\left(\frac{\rho(x\cos\alpha + y\sin\alpha)}{2\sqrt{2}},\right. \\
&\quad \left.\frac{\rho(y\cos\alpha - x\sin\alpha)}{2\sqrt{2}}\right),
\end{aligned} \tag{7}$$

where

$$P_k^{(\mu,\nu)}(t) = \frac{(-1)^k}{2^k k!}(1-t)^{-\mu}(1+t)^{-\nu}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[(1-t)^{k+\mu}(1+t)^{k+\nu}\right]$$

are Jacobi polynomials.

Let us define a new function family $\{\mathscr{G}_{n,m}(x, y|\alpha), n, m = 0, 1, \ldots\}$:

$$\begin{aligned}
\mathscr{G}_{n,m}(x,y|\alpha) &= \sum_{k=0}^{n+m} \mathrm{i}^k \cos^{n-k}\alpha \sin^{m-k}\alpha \\
&\quad \times P_k^{(n-k,m-k)}(-\cos 2\alpha)\mathscr{H}_{n+m-k,k}(x,y) \\
&= \mathrm{e}^{-x^2-y^2}\sum_{k=0}^{n+m} \mathrm{i}^k \cos^{n-k}\alpha \sin^{m-k}\alpha \\
&\quad \times P_k^{(n-k,m-k)}(-\cos 2\alpha) H_{n+m-k}\big(\sqrt{2}x\big) H_k\big(\sqrt{2}y\big).
\end{aligned} \tag{8}$$

Then the equality (7) may be rewritten as follows:

$$\begin{aligned}
&\int\!\!\int_{\mathbb{R}^2} \exp\left(-\mathrm{i}(x\xi + y\eta) + \frac{\mathrm{i}\psi(\xi,\eta,\alpha)}{\rho^2}\right) \\
&\quad \times \mathscr{H}_{n,m}\left(\frac{\xi}{\rho}, \frac{\eta}{\rho}\right) \mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad = \frac{\pi\rho^2}{\sqrt{2}}\left(\frac{1-\mathrm{i}}{\sqrt{2}}\right)^{n+m} \\
&\quad \times \exp\left(-\frac{\mathrm{i}\rho^2\psi(x,y,\alpha)}{8}\right) \\
&\quad \times \mathscr{G}_{n,m}\left(\frac{\rho(x\cos\alpha + y\sin\alpha)}{2\sqrt{2}},\right. \\
&\quad \left.\frac{\rho(y\cos\alpha - x\sin\alpha)}{2\sqrt{2}}\middle|\alpha\right).
\end{aligned} \tag{9}$$

Comparing the astigmatic transformation (9) for cases $\alpha = 0$ and $\pi/4$ with formulae (4) and (5), it is seen that

$$\mathscr{G}_{n,m}(x, y|0) = (-\mathrm{i})^m \mathscr{H}_{n,m}(x, y) \tag{10}$$

and

$$\mathscr{G}_{n,m}(x, y|\pi/4) = \begin{cases} (-1)^m 2^n m! \mathscr{L}_{m,n-m}(x, y) & (n \geqslant m), \\ (-1)^n 2^m n! \mathscr{L}_{n,m-n}(x, -y) & (n \leqslant m). \end{cases} \tag{11}$$

For this reason, we named $\mathscr{G}_{n,m}(x, y|\alpha)$ Hermite–Laguerre–Gaussian (HLG) beams.

Intermediate beams $\mathscr{G}_{n,m}(x, y|\alpha)$ keep many important features of HG and LG beams. First, every beam $\mathscr{G}_{n,m}(x, y|\alpha)$ is a product of $\mathrm{e}^{-x^2-y^2}$ and some polynomial in x, y of degree $n+m$. For example,

$$\begin{aligned}
\mathscr{G}_{0,0}(x,y|\alpha) &= \mathrm{e}^{-x^2-y^2}, \\
\mathscr{G}_{1,0}(x,y|\alpha) &= \mathrm{e}^{-x^2-y^2} 2\sqrt{2}(x\cos\alpha + \mathrm{i}y\sin\alpha), \\
\mathscr{G}_{0,1}(x,y|\alpha) &= \mathrm{e}^{-x^2-y^2} 2\sqrt{2}(x\sin\alpha - \mathrm{i}y\cos\alpha), \\
\mathscr{G}_{2,0}(x,y|\alpha) &= \mathrm{e}^{-x^2-y^2}\big(8(x\cos\alpha + \mathrm{i}y\sin\alpha)^2 - 2\cos 2\alpha\big), \\
\mathscr{G}_{1,1}(x,y|\alpha) &= \mathrm{e}^{-x^2-y^2}\big((4x^2 + 4y^2 - 2)\sin 2\alpha - 8\mathrm{i}xy\cos 2\alpha\big), \\
\mathscr{G}_{0,2}(x,y|\alpha) &= \mathrm{e}^{-x^2-y^2}\big(8(x\sin\alpha - \mathrm{i}y\cos\alpha)^2 + 2\cos 2\alpha\big).
\end{aligned}$$

Secondly, HLG beams are structurally stable under propagation and focusing because the sum of indices of each HG component, used in the expansion (8), is one and the same. Thirdly, and not so evidently, for any fixed α the family $\{\mathscr{G}_{n,m}(x, y|\alpha); n, m = 0, 1, \ldots\}$ is an orthogonal basis of the space $L_2(\mathbb{R}^2)$:

$$\int\!\!\int_{\mathbb{R}^2} \mathscr{G}_{n,m}(x,y|\alpha)\overline{\mathscr{G}}_{N,M}(x,y|\alpha)\,\mathrm{d}x\,\mathrm{d}y = 2^{n+m-1}\pi n! m! \delta_{nN}\delta_{mM}. \tag{12}$$

Here and below, an overline means complex conjugation. It is interesting to note that the norm of an HLG beam, $\|\mathscr{G}_{n,m}(x, y|\alpha)\| = \sqrt{2^{n+m-1}\pi n! m!}$, has no dependence on α. This is a consequence of the energy conservation law applying to a Fourier based transformation (9).

To prove the equality (12) we write the transformation (9) in the form

$$\begin{aligned}
\mathscr{G}_{n,m}(x,y|\alpha) &= \frac{\sqrt{2}}{\pi}\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^{n+m}\exp(\mathrm{i}x^2 - \mathrm{i}y^2) \\
&\quad \times \int\!\!\int_{\mathbb{R}^2} \exp\big(\mathrm{i}\psi(\xi,\eta,\alpha) - 2\sqrt{2}\mathrm{i}\xi(x\cos\alpha - y\sin\alpha) \\
&\quad - 2\sqrt{2}\mathrm{i}\eta(y\cos\alpha + x\sin\alpha)\big)\mathscr{H}_{n,m}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned} \tag{13}$$

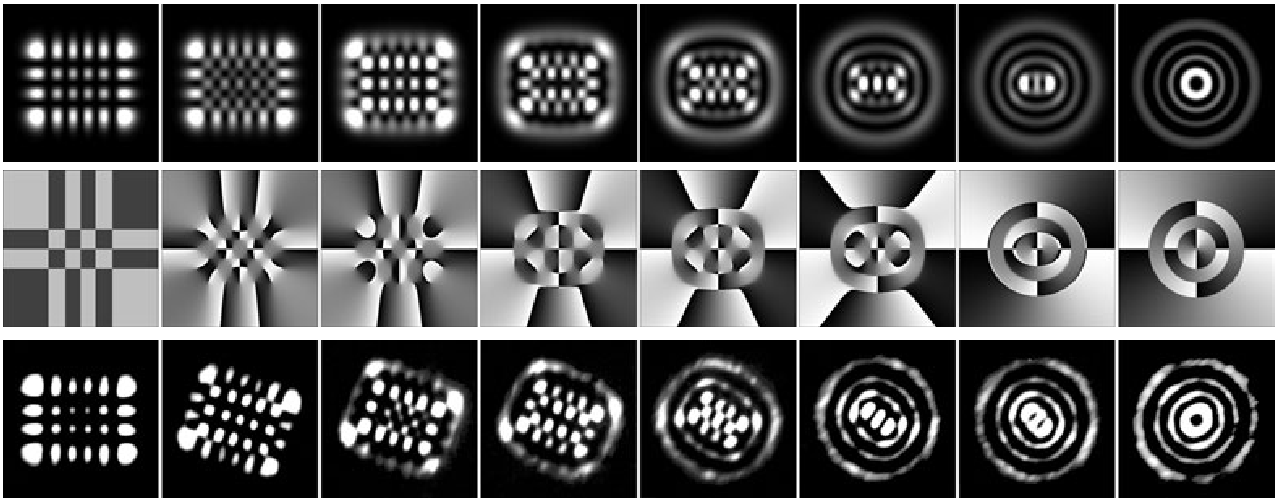

Figure 1. Numerical simulation and experimental realization of HLG beams $\mathscr{G}_{5,3}(x, y|\alpha)$ for $\alpha \in [0, \pi/4]$. The top and middle rows present intensity and phase theoretical distributions. The black colour corresponds to zero intensity and zero phase, whereas the white colour corresponds to maximal intensity and phase 2π. A sharp black–white phase jump corresponds to a lacing of 0 and 2π phases. The bottom row presents experimentally registered intensities. The experimental set-up is the usual cylindrical lens mode converter of HG beams into LG beams [2], mentioned above. The transformation (9) is realized by the rotation of cylindrical lenses with respect to an input HG beam.

The generating function for Hermite polynomials,

$$e^{-s^2+2sx} = \sum_{k=0}^{\infty} H_k(x)\frac{s^k}{k!},$$

helps to find the generating function for HLG beams:

$$\begin{aligned} G(x, y, \alpha, s, t) &= \sum_{n,m=0}^{\infty} \mathscr{G}_{n,m}(x, y|\alpha)\frac{s^n t^m}{n!m!} \\ &= \exp\big(-x^2 - y^2 - \psi(s, t, \alpha) + 2\sqrt{2}x(s\cos\alpha + t\sin\alpha) \\ &\quad + 2\sqrt{2}\mathrm{i}y(s\sin\alpha - t\cos\alpha)\big). \end{aligned} \tag{14}$$

Then

$$\begin{aligned} &\sum_{n,m=0}^{\infty}\sum_{N,M=0}^{\infty}\left(\int\!\!\int_{\mathbb{R}^2} \mathscr{G}_{n,m}(x, y|\alpha)\overline{\mathscr{G}}_{N,M}(x, y|\alpha)\,\mathrm{d}x\,\mathrm{d}y\right) \\ &\quad\times \frac{s^n t^m}{n!m!}\frac{S^N T^M}{N!M!} \\ &= \int\!\!\int_{\mathbb{R}^2} G(x, y, \alpha, s, t)\overline{G}(x, y, \alpha, S, T)\,\mathrm{d}x\,\mathrm{d}y \\ &= \exp\big(-\psi(s, t, \alpha) - \psi(S, T, \alpha)\big) \\ &\quad\times \int_{\mathbb{R}} \exp\big(-2x^2 + 2\sqrt{2}x[(s+S)\cos\alpha \\ &\quad + (t+T)\sin\alpha]\big)\,\mathrm{d}x \\ &\quad\times \int_{\mathbb{R}} \exp\big(-2y^2 + 2\sqrt{2}\mathrm{i}y[(s-S)\sin\alpha \\ &\quad - (t-T)\cos\alpha]\big)\,\mathrm{d}y \\ &= \frac{\pi}{2}\exp(2sS + 2tT) \\ &= \frac{\pi}{2}\sum_{n,m=0}^{\infty}\frac{(2sS)^n(2tT)^m}{n!m!}. \end{aligned} \tag{15}$$

The equality (12) follows from the comparison of the coefficients of first and the last series.

Applying a generating function technique, various formulae with HLG beams may be found, for example, recurrent relations:

$$\begin{aligned} 2\sqrt{2}x\mathscr{G}_{n,m} &= \cos\alpha\mathscr{G}_{n+1,m} + \sin\alpha\mathscr{G}_{n,m+1} \\ &\quad + 2n\cos\alpha\mathscr{G}_{n-1,m} + 2m\sin\alpha\mathscr{G}_{n,m-1}, \\ 2\sqrt{2}\mathrm{i}y\mathscr{G}_{n,m} &= \sin\alpha\mathscr{G}_{n+1,m} - \cos\alpha\mathscr{G}_{n,m+1} \\ &\quad - 2n\sin\alpha\mathscr{G}_{n-1,m} + 2m\cos\alpha\mathscr{G}_{n,m-1}; \end{aligned} \tag{16}$$

derivative relations:

$$\begin{aligned} \frac{\partial\mathscr{G}_{n,m}}{\partial x} &= 2x\mathscr{G}_{n,m} - \sqrt{2}\cos\alpha\mathscr{G}_{n+1,m} - \sqrt{2}\sin\alpha\mathscr{G}_{n,m+1}, \\ \frac{\partial\mathscr{G}_{n,m}}{\partial y} &= 2y\mathscr{G}_{n,m} + \mathrm{i}\sqrt{2}\sin\alpha\mathscr{G}_{n+1,m} - \mathrm{i}\sqrt{2}\cos\alpha\mathscr{G}_{n,m+1}, \\ \frac{\partial\mathscr{G}_{n,m}}{\partial\alpha} &= m\mathscr{G}_{n+1,m-1} - n\mathscr{G}_{n-1,m+1}; \end{aligned} \tag{17}$$

and integral moments:

$$\begin{aligned} &\int\!\!\int_{\mathbb{R}^2} x|\mathscr{G}_{n,m}|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int\!\!\int_{\mathbb{R}^2} y|\mathscr{G}_{n,m}|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int\!\!\int_{\mathbb{R}^2} xy|\mathscr{G}_{n,m}|^2\,\mathrm{d}x\,\mathrm{d}y = 0, \\ &\int\!\!\int_{\mathbb{R}^2} (x^2 + y^2)|\mathscr{G}_{n,m}|^2\,\mathrm{d}x\,\mathrm{d}y = \frac{n+m+1}{2}\,\|\mathscr{G}_{n,m}\|^2, \\ &\int\!\!\int_{\mathbb{R}^2} (x^2 - y^2)|\mathscr{G}_{n,m}|^2\,\mathrm{d}x\,\mathrm{d}y = \frac{n-m}{2}\,\|\mathscr{G}_{n,m}\|^2\cos 2\alpha. \end{aligned} \tag{18}$$

Here we used a short notation $\mathscr{G}_{n,m}$ for an HLG beam $\mathscr{G}_{n,m}(x, y|\alpha)$.

Many features of HG beams, LG beams and intermediate HLG beams are common. However, there is an important distinction. Both HG and LG beams, excepting trivial cases, have zero lines, but HLG beams do not. A presence of zero lines for HG beams is a corollary of their real-valued structure. LG beams are complex-valued, and the existence of zero lines for them is a result of the presence of a Laguerre polynomial factor. In the case of intermediate HLG beams, all zeros are isolated points. This is the general situation for almost all complex-valued functions. Isolated zeros are points of intersection of real and imaginary parts of these functions. Such isolated zeros are often named singular points or phase singularities [6]. Currently, singular optics is a field of intensive investigation [7, 8]. To classify all possible zeros is a difficult problem (see, for example, [9, 10]). We prefer to use a somewhat simpler but less detailed description. Namely, if a phase circulation around an isolated zero in the counterclockwise direction is positive, then we call such a zero positive, or a zero of z type. If the circulation is negative, then this is a negative zero, or a zero of $\bar{z}$ type.

Intermediate HLG beams have isolated zeros of both types. An example of $\mathscr{G}_{n,m}(x, y|\alpha)$ is shown in figure 1. For $\alpha = 0$ an HLG beam is an HG beam; its phase has only two values, 0 and π, and the beam zeros are straight lines. For $\alpha > 0$ an HLG beam has real and imaginary components. Intersection points of zero lines of both components are beam isolated zeros. A zero motion, as the angle α is changed, is rather complicated. Some of the zeros of opposite sign annihilate, but zeros of the same sign stick together and finally for $\alpha = \pi/4$ an LG beam is obtained.

One more result on HLG beams is connected with the expansion (8). Substituting (10) into equality (8), we get

$$\mathscr{G}_{n,m}(x, y|\alpha) = \sum_{k=0}^{n+m} (-1)^k \cos^{n-k}\alpha \sin^{m-k}\alpha \times P_k^{(n-k,m-k)}(-\cos 2\alpha)\mathscr{G}_{n+m-k,k}(x, y|0), \tag{19}$$

and for $\alpha = \pi/4$ three various orthogonal polynomial families (Hermite, Laguerre, and Jacobi) meet together:

$$\mathscr{G}_{n,m}(x, y|\pi/4) = 2^{-(n+m)/2} \times \sum_{k=0}^{n+m} (-2)^k P_k^{(n-k,m-k)}(0)\mathscr{G}_{n+m-k,k}(x, y|0). \tag{20}$$

The expansion (20) in various formulations has been found and discussed in many papers (see, for example, [2, 5, 11]). However, much more general HLG expansions exist. It may be found, if we consider Jacobi polynomials as a part of Wigner d-functions, i.e. functions that are used for the description of a rotation in three-dimensional space. Namely:

$$\mathscr{G}_{n,m}(x\cos\beta - y\sin\beta, y\cos\beta + x\sin\beta|\theta) = \sum_{k=0}^{n+m} \lambda_k^{(n,m)} \mathscr{G}_{n+m-k,k}(x, y|\alpha), \tag{21}$$

where

$$\lambda_k^{(n,m)} = (-1)^k \big(\cos\beta\cos(\theta-\alpha) + \mathrm{i}\sin\beta\sin(\theta+\alpha)\big)^{n-k} \times \big(\cos\beta\sin(\theta-\alpha) - \mathrm{i}\sin\beta\cos(\theta+\alpha)\big)^{m-k} \times P_k^{(n-k,m-k)}\big(\sin^2\beta\cos 2(\theta+\alpha) - \cos^2\beta\cos 2(\theta-\alpha)\big). \tag{22}$$

Angular momentum properties of HLG beams are also a subject of interest. It is known that, in general, a light field carries some angular momentum which may be transferred to a captured microparticle, causing its rotation or motion in a predetermined trajectory. The angular momentum of a coherent light field $F(x, y)$ with frequency ω is defined as follows [12, 13]:

$$L[F] = \frac{1}{E}\int\!\!\int_{\mathbb{R}^2} M(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{23}$$

where

$$E = \int\!\!\int_{\mathbb{R}^2} |F(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y \tag{24a}$$

and

$$M(x, y) = \frac{1}{\omega}\,\mathrm{Im}\left[\overline{F}\left(x\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial x}\right)\right] \tag{24b}$$

are the beam energy and the beam angular momentum density respectively. HG beams have no angular momentum, but LG beams do. It is interesting to find the angular momentum $L\big[\mathscr{G}_{n,m}(x, y|\alpha)\big]$ for any α. Using (12), (16)–(18), we get

$$L\big[\mathscr{G}_{n,m}(x, y|\alpha)\big] = \frac{n-m}{\omega}\sin 2\alpha. \tag{25}$$

As is noted above, there is a residual astigmatism in some optical schemes for astigmatic transformation. Let usconsider its influence on an HLG beam angular momentum. The presence of some phase perturbation $\varphi(x, y)$ in the field $F(x, y)$ changes its angular momentum in the following way:

$$\begin{aligned} L[\mathrm{e}^{\mathrm{i}\varphi}F] &= \frac{1}{\omega E}\,\mathrm{Im}\int\!\!\int_{\mathbb{R}^2}\left[\mathrm{e}^{-\mathrm{i}\varphi}\overline{F}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(\mathrm{e}^{\mathrm{i}\varphi}F)\right]\mathrm{d}x\,\mathrm{d}y \\ &= L[F] + \frac{1}{\omega E}\int\!\!\int_{\mathbb{R}^2}|F|^2\left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{26}$$

For an astigmatic influence $\varphi(x, y) = \psi(x, y, \beta)$ and an HLG beam we obtain:

$$L\left[\mathrm{e}^{\mathrm{i}\psi(x,y,\beta)}\mathscr{G}_{n,m}(x, y|\alpha)\right] = \frac{n-m}{\omega}(\sin 2\alpha + \cos 2\alpha\sin 2\beta). \tag{27}$$

In particular, for HG and LG beams the equality (27) reduces to the following:

$$\begin{aligned} L\big[\mathrm{e}^{\mathrm{i}\psi(x,y,\beta)}\mathscr{G}_{n,m}(x, y|0)\big] &= \frac{n-m}{\omega}\sin 2\beta, \\ L\big[\mathrm{e}^{2\mathrm{i}xy}\mathscr{G}_{n,m}(x, y|0)\big] &= \frac{n-m}{\omega}, \\ L\big[\mathrm{e}^{\mathrm{i}\psi(x,y,\beta)}\mathscr{G}_{n,m}(x, y|\pi/4)\big] &= \frac{n-m}{\omega}. \end{aligned} \tag{28}$$

So, the angular momentum of an HG beam depends on the astigmatic influence, but the angular momentum of an LG beam cannot be changed by the astigmatic influence. Therefore, optical schemes with or without residual astigmatism produce LG beams with one and the same angular momentum.

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